Deformation quantization of linear dissipative systems

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Abstract

A simple pseudo-Hamiltonian formulation is proposed for the linear inhomogeneous systems of ODEs. In contrast to the usual Hamiltonian mechanics, our approach is based on the use of a non-stationary Poisson bracket, i.e. the corresponding Poisson tensor is allowed to explicitly depend on time. Starting from this pseudo-Hamiltonian formulation we develop a consistent deformation quantization procedure involving a non-stationary star-product $*_t$ and an "extended" operator of time derivative $D_t = \partial_t + \cdots$, differentiating the $*_t$-product. As in the usual case, the $*_t$-algebra of physical observables is shown to admit an essentially unique (time dependent) trace functional Tr_t. Using these ingredients we construct a complete and fully consistent quantum-mechanical description for any linear dynamical system with or without dissipation. The general quantization method is exemplified by the models of damped oscillator and radiating point charge.

1 Introduction

The problem of quantum-mechanical description of dissipative systems remains for decades a recurrently discussed physical topic with a number of important applications (see [1]-[17] and references therein). It has also some theoretical importance as a touchstone for testing various quantization methods. Our interest to the problem is inspired by recent developments in deformation quantization [18]. In this paper we consider the problem of deformation quantization for dissipative systems, whose classical dynamics is specified by linear inhomogeneous systems of ODEs.

In spite of a large number of papers devoted to the quantum-mechanical treatment of various dissipative systems there is no commonly accepted definition of the dissipation phenomenon itself. It seems that the most characteristic property of the dissipation, shared by all the systems known as dissipative, is the presence of *attractors* [19]. Recall that an attractor $A \subset M$ is a compact, measure-zero subset in the phase space of the system M, possessing the property of being limiting set for any trajectory passing through a sufficiently small neighborhood of A. In a simple situation

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the attractor is just a stable fixed point (sink) or a closed curve (limit cycle); in the general case, however, much more complicated dissipative structures may occur (e.g. strange attractors).

From the Liouville theorem about the conservation of phase-space volume [20] it follows immediately that no smooth compatible symplectic structure can exist in the vicinity of an attractor: As the measure of attractor is zero, the Liouville volume form, given by the Pfaffian of the symplectic 2-form, becomes necessarily infinite at the points of the attractor. The attractors are thus obstructions for constructing the conventional Hamiltonian description of the dissipative systems and this is also the reason why the terms "dissipative" and "non-Hamiltonian" are often used as synonyms.

In principle, the requirement of smoothness of the symplectic structure is not so crucial for deformation quantization as only the Poisson bracket and a trace density are actually used. The problem, however, is that even for a linear dissipative system with one-point attractor (e.g. focus) the corresponding Poisson bracket appears to be highly nonlinear, making impossible any practical calculations [21] (see also [27] for similar discussion of Kepler's problem).

To get round the "no-go theorem" above and obtain a practicable quantization scheme we allow the symplectic structure to depend on time explicitly. The idea is as follows: Since the phase-space trajectories reach the attractor A only asymptotically (as $t \to \infty$) one can try to construct a time dependent symplectic structure $\Omega(t)$ which would be a smooth function of time and such that $\lim_{t\to\infty} \Omega(t) = \infty$. We show that such a non-stationary symplectic structure does exist for any linear dynamical system and can be explicitly constructed by the fundamental matrix of the linear system. Moreover, for one-dimensional dynamical systems, described by a linear second-order equation, one can always find a time-independent Ω, whereas for the general multi-dimensional system such choice is impossible. In this respect the one-dimensional dissipative systems, like the damped linear oscillator, are not indicative examples.

The unavoidable time dependence of multi-dimensional symplectic structure has further consequences for nearly Hamiltonian systems, i.e. the systems whose equations of motion have the form of a small perturbation over the Hamiltonian ones. No matter how small the value of perturbation is, the perturbed equations may no longer be Hamiltonian, that indicates presence of genuine dissipation. It is still possible, however, to construct the first-order action functional if one admits a non-stationary symplectic structure; in so doing, the explicit time dependence of Ω can violate any phase-space polarization [22](e.g., this may admit no separation of the variables onto position coordinates and conjugate momenta) making impossible the passage to the second-order Lagrangian formalism. The absence of the second-order Lagrangian description for some classical systems with two or more degrees of freedom was recognized long ago [23], [24], [25]. For a nearly Hamiltonian system it is reasonable to require the Lagrangian, if any, to pass into the non-perturbed one as the parameter of dissipation (e.g. friction constant) vanishes. The last requirement imposes very strong restriction on physically reasonable Lagrangians. For example, in [21] we have described all the second-order Lagrangians for the reduced Lorentz-Dirac equation in the case of homogeneous magnetic field, but none of them reproduces the free particle motion as the charge of the particle tends to zero. The last fact shows clearly the advantage of the first-order formalism over the second-order one. For the general discussion on peculiarities of the inverse problem of variational calculus in the first-order formalism we refer the reader to [26].

The absence of a phase-space polarization favors also the use of deformation quantization over the canonical quantization procedure. Being the function of time, but not of the phase-space coordinates, the corresponding Poisson bracket can easily be quantized by the usual Weyl-Moyal formula giving rise to a *non-stationary* star-product $*_t$. It is the point where our approach deviates from the conventional scheme of deformation quantization [27], [28]. Notice that in the non-stationary case the usual time derivative does not differentiate the $*_t$-product of quantum

observables; instead one can define an extended time derivative $D_t = \partial_t + \cdots$ that would be compatible with the $*_t$-product in the sense of the Leibnitz rule. As in the usual case, the $*_t$-algebra of quantum observables is shown to admit an essentially unique (non-stationary) trace functional $\mathrm{Tr_t}$. Using these ingredients, we define the quantum Liouville equation governing the evolution of quantum-mechanical states, as well as the rule for computing the expectation values of physical observables. As a result we get a complete quantum-mechanical description for any linear dynamical system. Moreover, the physical content of the theory is shown to be independent of any ambiguities concerning the choice of the non-stationary symplectic structure $\Omega(t)$ as the quantum Liouville equation coincides precisely with the classical one (see Proposition 1). Notice that a lot of objections have been published for many years against possibility to consistently quantize such systems in the Hamiltonian framework, even in principle, see e.g. [14]-[17].

By way of illustration we consider the quantization problem for the damped linear oscillator and for the radiating point charge moving in a homogeneous magnetic field. In both cases the quantum dynamics seems very reasonable. In particular, the time evolution of mean energy, defined in terms of the corresponding unperturbed system, is shown to coincide with evolution of the classical energy.

2 Pseudo-Hamiltonian formulation of linear dynamical systems

We start with an inhomogeneous linear system of ODEs

$$\dot{x}^i = A^i_j(t)x^j + J^i(t) \tag{1}$$

defined on a linear phase space with coordinates x^i. Hereafter the overdot stands for the derivative with respect to time t. When A and J are independent of time, one speaks of an autonomous system of ODEs. In this paper we are interested in linear dynamical systems which are a (small) perturbation of a Hamiltonian one. Although the perturbed system may no longer be a Hamiltonian one[1], the number of degrees of freedom remains the same. In particular, we will always assume the phase space of the system (1) to be even dimensional, i.e. $i = 1, ..., 2n$.

Our first observation is that any such system can be derived from the variation principle for a quadratic action functional if the explicit time dependence is admitted in the integrand. Consider the following ansatz:

$$S[x] = \frac{1}{2}\int dt(x^i\Omega_{ij}(t)\dot{x}^j - x^iB_{ij}(t)x^j - 2C_i(t)x^i)\,, \tag{2}$$

where

$$\Omega_{ij} = -\Omega_{ji}\,, \qquad B_{ij} = B_{ji}\,, \qquad \det(\Omega_{ij}) \neq 0\,. \tag{3}$$

Structurally, the functional S is similar to the first-order action associated with the Hamiltonian

$$H = \frac{1}{2}x^iB_{ij}(t)x^j - C_i(t)x^i\,, \tag{4}$$

but unlike the usual Hamiltonian formalism we allow the symplectic form Ω to depend on time. This assumption appears to be crucial for the description of multi-dimensional dissipative systems as will be seen below.

[1]In this case, the perturbation can't be induced by a perturbation of the Hamiltonian.

Varying this action functional, we come to the following equations [2]:

$$\frac{\delta S}{\delta x^i} = 0 \quad \Leftrightarrow \quad \dot{x} = \Omega^{-1}\left(B - \frac{1}{2}\dot{\Omega}\right)x + \Omega^{-1}C\,. \tag{5}$$

In order for these equations to be equivalent to the original ones (1) we must set

$$A = \Omega^{-1}\left(B - \frac{1}{2}\dot{\Omega}\right), \quad J = \Omega^{-1}C\,, \tag{6}$$

or, what is the same,

$$\frac{1}{2}\dot{\Omega} = B - \Omega A\,, \qquad C = \Omega J\,. \tag{7}$$

Decomposing the first matrix equation onto symmetric and anti-symmetric parts, we finally get

$$\dot{\Omega} = -(\Omega A + A^t\Omega)\,, \qquad B = \frac{1}{2}\Big(\Omega A - A^t\Omega\Big)\,, \qquad C = \Omega J\,, \tag{8}$$

A^t being the transpose of the matrix A. Only the first relation is nontrivial (it is a linear ODE on Ω), while the other two are just definitions of the matrices B and C.

Recall that the square matrix $\Gamma(t)$ is called the fundamental solution to the system (1) if

$$\dot{\Gamma} = A\Gamma\,, \qquad \Gamma(0) = 1\,. \tag{9}$$

The columns of this matrix constitute the basis in the linear space of solutions to Eqs.(1). Given the matrix Γ, the general solution to the first equation (8) can be written as

$$\Omega = \Lambda^t\Omega_0\Lambda\,, \tag{10}$$

where $\Lambda = \Gamma^{-1}$, and $\Omega_0 = -\Omega_0^t$ is a constant non-degenerate matrix. The matrix Ω_0 encodes all the ambiguity in the definition of quadratic action functional (2) for the given system of ODEs (1).

In the autonomous case, the system (1) is explicitly integrable in elementary functions and hence the action (2) can be written in a closed form. By way of illustration let us consider two physical examples: the damped linear oscillator, and the non-relativistic motion of a point charge with a due regard for the radiation back reaction.

Example 1. The equation describing the linear oscillator with friction reads

$$\ddot{x} + 2\alpha x + \omega^2 x = 0\,. \tag{11}$$

Here ω is the frequency and $\alpha \geq 0$ is the coefficient of friction. Introducing the auxiliary variable

$$p = \frac{\dot{x} + \alpha x}{\sqrt{1 - \alpha^2/\omega^2}}\,,$$

one can replace (11) with the following pair of first-order equations:

$$\dot{x} = p\sqrt{1 - \alpha^2/\omega^2} - \alpha x\,, \qquad \dot{p} = -\omega^2 x\sqrt{1 - \alpha^2/\omega^2} - \alpha p\,. \tag{12}$$

[2] Here we use the matrix notation.

According to (2) and (8), the action functional to this system is given by

$$S[x,p] = c\int dt\left(p\dot{x} - \frac{1}{2}\sqrt{1-\alpha^2/\omega^2}(p^2+\omega^2x^2) + \alpha px\right)e^{2\alpha t}\,, \tag{13}$$

c being an arbitrary constant. In the regime of aperiodic damping ($\alpha > \omega$) the action functional becomes complex.

In this simple case it is also possible to construct a first-order action functional involving the canonical symplectic structure. For example, varying the action

$$S[x,p] = \int dt\left(p\dot{x} - \frac{1}{2}(e^{-\alpha t}p^2 + \omega^2 e^{\alpha t}x^2)\right)\,, \tag{14}$$

one gets the Hamiltonian equations

$$\dot{x} = e^{-\alpha t}p\,, \qquad \dot{p} = -\omega^2 e^{\alpha t}x\,, \tag{15}$$

which are obviously equivalent to Eq.(11). Notice that, contrary to the system (12), the stationary point $x = p = 0$ of (15) is not an attractor.

It is significant that both (13) and (14) come to the standard Hamiltonian action for the harmonic oscillator when $\alpha \to 0$. Since the action functional carries all the information about classical and quantum dynamics, the last fact makes possible a consistent interpretation of friction as a small perturbation over the given Hamiltonian system, rather than something leading to a completely different physical system.

Example 2. The effective dynamics of a non-relativistic charged particle is governed by the Lorentz equation [29]

$$m\ddot{\mathrm{x}} = e\mathbf{E} + \frac{e}{c}[\dot{\mathrm{x}},\mathbf{H}] + \frac{2e^2}{3c^3}\,\dddot{\mathbf{x}}\ . \tag{16}$$

Here $\mathrm{x}(t) \in \mathbb{R}^3$ is a trajectory of the particle, $\mathbf{E}$ and $\mathbf{H}$ are 3-vectors of electric and magnetic fields, and the constants c and e denote the light velocity and electric charge of the particle. As is seen, the Lorentz equation involves third time derivative of the trajectory. It is the term which describes the back reaction of the radiation emitted by the accelerating charge. Since the order of the equation is greater than two, it cannot be assigned with a straightforward mechanical interpretation: in the realm of Newtonian mechanics, a trajectory of a scalar particle is uniquely specified by initial position and velocity. Besides, together with physically meaningful solutions Eq.(16) allows a set of nonphysical ones [29]. It turns out that both mentioned problems can be resolved by means of the *reduction of order procedure* (see e.g. [21], [29], [30]). Namely, Eq. (16) is replaced by a second-order equation $\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x},\dot{\mathbf{x}},e)$ such that all the solutions to the latter would solve the former. The last requirement leads to a partial differential equation on the function $\mathbf{g}(\mathbf{x},\mathbf{v},e)$ having a unique solution with $\mathbf{g}(\mathbf{x},\mathbf{v},0) = 0$.

Consider for example a non-relativistic particle moving in a homogeneous magnetic field, i.e. $\mathbf{E} = 0$, $\mathbf{H} = (0,0,H)$ and $H = const$. The reduced Lorentz equation has the form [21]

$$\ddot{x} = A\dot{x} - B\dot{y}\,, \qquad \ddot{y} = B\dot{x} + A\dot{y}\,, \qquad \ddot{z} = 0\,,$$

where $\mathbf{x} = (x,y,z)$ and

$$A = \frac{6-\sqrt{6}\sqrt{3+\sqrt{9+64e^6H^2}}}{8e^2} \approx -\frac{2}{3}e^4H^2\,, \qquad B = \frac{eH\sqrt{6}}{\sqrt{3+\sqrt{9+64e^6H^2}}} \approx eH\,. \tag{17}$$

Here we have set $m = c = 1$. Since the evolution along z represents the free motion and decouples from the dynamics in the xy-plane, we restrict our consideration to the first two equations.

Setting formally $A = 0$, we arrive at the usual Lorentz equations describing the motion of a charge in response to the "effective" magnetic field $\mathbf{B} = (0, 0, B/e)$. In this case, the trajectories are concentric circles. For $A \neq 0$ the particle spirals at the origin of xy-plane. So, it is natural to regard A as the coefficient of friction.

In order to construct an action functional for the second-order Eqs.(17), we replace them with an equivalent system of first-order ones. Let us choose the auxiliary variables as

$$p = \dot{x} + \frac{B}{2}y, \qquad q = \dot{y} - \frac{B}{2}x\,. \tag{18}$$

Then

$$\begin{aligned} \dot{x} &= p - \frac{B}{2}y\,, & \dot{y} &= q + \frac{B}{2}x\,, \\ \dot{p} &= -\frac{B}{2}q - \frac{B^2}{4}x + A\left(p - \frac{B}{2}y\right)\,, & \dot{q} &= \frac{B}{2}p - \frac{B^2}{4}y + A\left(q + \frac{B}{2}x\right)\,. \end{aligned} \tag{19}$$

Applying the general formulas (2) and (8), we arrive at the following expression for the first-order action functional:

$$\begin{aligned} S[x,y,q,p] = \frac{1}{4(A^2+B^2)}\int dt e^{-At}[2a(t)(p\dot{x} - x\dot{p} + q\dot{y} - y\dot{q}) \\ +2b(t)(q\dot{x} - x\dot{q} + y\dot{p} - p\dot{y}) + 2c(t)(p\dot{q} - q\dot{p}) + 2d(t)(x\dot{y} - y\dot{x}) \\ +e(t)(p^2+q^2) + f(t)(x^2+y^2) + g(t)(px+qy) + j(t)(qx-py)]\,, \end{aligned} \tag{20}$$

where

$$a(t) = A^2\cos(Bt) + \frac{1}{2}B^2(e^{-At} + e^{At})\,, \quad b(t) = A^2\sin(Bt) - ABe^{At} + AB\,\cos(Bt)\,,$$

$$c(t) = e^{-At}B + 2A\,\sin(Bt)\,, \quad d(t) = \frac{1}{4}B^3(e^{-At} - e^{At}) - \frac{1}{2}B^2A\,\sin(Bt) + A^2B(\cos(Bt) - e^{At})\,,$$

$$e(t) = e^{-At}B^2 + A^2\cos(Bt) + A\,\sin(Bt)B\,,$$

$$f(t) = \frac{1}{4}B\left(B^3e^{-At} + BA^2\cos(Bt) - A\,\sin(Bt)[B^2 + 2A^2]\right)\,,$$

$$g(t) = -A\,\cos(Bt)B^2 - A^3\cos(Bt)\,, \quad j(t) = -A^3\sin(Bt) + e^{-At}B^3 + A^2B\,\cos(Bt)\,.$$

In the limit of zero friction $A \to 0$, this action tends to the usual action for the charged particle in a homogeneous magnetic field B.

Notice that the symplectic structure entering the Lagrangian (20) does not possess xy-polarization (i.e. the 2-form Ω does not vanish upon restriction on the 2-plane $x = const, y = const$) when $A \neq 0$. This makes impossible the algebraic elimination of p and q from the action (24) and thus obtaining a second-order action in terms of x and y. The latter fact agrees well with the general statement of Ref.[21] about nonexistence of a second-order action functional for Eq. (17), which would pass to the standard action functional for a free particle when $e \to 0$.

It can also be shown that unlike the previous example, the system (17) does not admit a first-order action involving a stationary symplectic structure and having the standard free limit. The unavoidable time dependence of the symplectic structure may be thus viewed as a specific feature of multi-dimensional dissipative systems.

Using the Hamiltonian (4) and the symplectic 2-form Ω it is possible to rewrite Eqs.(1) in a pseudo-Hamiltonian form. To this end we introduce the following nonstationary Poisson brackets:

$$\{F, G\}_t = \Pi^{ij}(t)\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j}\,, \qquad \Pi = \Omega^{-1}\,, \tag{21}$$

where F and G are functions of the phase-space coordinates x^i and the time t. Clearly, these brackets satisfy all the properties of the Poisson brackets: bi-linearity, skew-symmetry, the Leibnitz and Jacobi identities. In accordance with (5) the evolution of an arbitrary physical observable $F(t, x)$ is described by the following equation:

$$\frac{dF}{dt} = D_t F + \{F, H\}_t\,, \tag{22}$$

where

$$H = \frac{1}{2}x^i B_{ij} x^j + C_i(t)x^i \tag{23}$$

and

$$D_t F \equiv \frac{\partial F}{\partial t} - \frac{1}{2}x^i\dot{\Omega}_{ik}\Pi^{kj}\frac{\partial F}{\partial x^j} = \frac{\partial F}{\partial t} - \frac{1}{2}x^i\dot{\Omega}_{ik}\{x^k, F\}_t \tag{24}$$

is the "extended" partial derivative in t. When Ω is constant, the last term in (24) vanishes and we arrive at the conventional Hamiltonian equations w.r.t. the canonical Poisson brackets and the Hamiltonian (23). The main property of the extended time derivative D_t is that it differentiates the non-stationary Poisson brackets (21), i.e.

$$D_t\{F, G\}_t = \{D_t F, G\}_t + \{F, D_t G\}_t\,, \tag{25}$$

for any $F(t, x)$ and $G(t, x)$. Using this property one can deduce the Poisson theorem:

$$\frac{dF}{dt} = 0\,, \quad \frac{dG}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\{F, G\}_t = 0 \tag{26}$$

(the Poisson bracket of two conserved quantities is conserved).

In the full analogy with the conventional Hamiltonian mechanics, a state of the system is described by a classical distribution function $\rho_{cl}(x, t)$ subject to the normalization condition

$$\int d\mu\rho_{cl}(x, t) = 1\,, \tag{27}$$

$d\mu \equiv \sqrt{\det\Omega}\, d^{2n}x$ being the Liouville measure associated to the non-stationary symplectic form Ω. The time dependence of $\rho_{cl}(x, t)$ is determined by the modified Liouville equation (cf. Eq. (22))

$$D_t\rho_{cl} = \{H, \rho_{cl}\}_t\,. \tag{28}$$

Note that the time evolution preserves the normalization condition (27). Indeed, using the obvious identity

$$\int d\mu\{F, G\}_t = 0\,, \tag{29}$$

where one of the functions $F(x)$ and $G(x)$ vanishes on infinity, one can find

$$\frac{d}{dt}\int d\mu\rho_{cl}(x, t) = \left(\frac{d\ln\Delta}{dt} - \dot{\Omega}_{ij}\Pi^{ij}\right)\int d\mu\rho_{cl}(x, t) = 0\,, \qquad \Delta = \sqrt{\det\Omega}\,. \tag{30}$$

The *pure states* of the classical system correspond to the δ-distributions

$$\rho_{cl}(x,t) = \Delta^{-1}(t)\delta^{2n}(x - x_0(t))\,, \tag{31}$$

supported on the integral trajectories $x_0(t)$ of the system (1).

As the final remark, let us note that the above pseudo-Hamiltonian formalism is applicable for arbitrary (not necessary quadratic) Hamiltonians and can easily be derived/justified in the formalism of constrained dynamics [31] applied to the first-order action (2).

3 Deformation quantization of pseudo-Hamiltonian systems

Example 2 of Sec.1 shows that the most characteristic feature of multi-dimensional dissipative systems is the lack of a phase-space polarization compatible with dynamics: Though at each time moment t_0 one can split the phase-space variables $x^i = (q^a, p_b)$ on "coordinates" and "momenta" satisfying the canonical Poisson bracket relations

$$\{q^a, q^b\}_{t_0} = 0\,, \quad \{p_a, p_b\}_{t_0} = 0\,, \quad \{q^a, p_b\}_{t_0} = \delta^a_b\,, \tag{32}$$

these relations may not hold true at the next time moment. This is due to the explicit (and in many interesting cases unavoidable) time dependence of the symplectic form Ω entering the first-order action (2). The absence of a natural polarization favors the use of deformation quantization over the canonical quantization procedure.

In the approach of deformation quantization the classical observables (i.e. functions in the phase-space variables) are identified with symbols of operators [27], [28]; in so doing, the pointwise multiplication of functions is replaced by an associative noncommutative star-product. Using the non-stationary Poisson bracket (21), we define the time dependent $*_t$-product by the usual Weyl-Moyal formula

$$(F *_t G)(x,t) \equiv \exp\left(\frac{i\hbar}{2}\Pi^{ij}(t)\frac{\partial^2}{\partial x^i \partial y^j}\right) F(t,x)G(t,y)|_{x=y} = F\cdot G + \frac{i\hbar}{2}\{F,G\}_t + O(\hbar^2)\,, \tag{33}$$

It is clear that the time dependence of Π does not affect the associativity of the Weyl-Moyal star-product, so we have

$$(F *_t G) *_t H = F *_t (G *_t H)\,, \qquad \forall\, F, G, H \in C^\infty(\mathbb{R}^{2n})\,. \tag{34}$$

In order to define the notion of a quantum state we endow the $*_t$-algebra with the following trace functional:

$$\mathrm{Tr}_t(F) = \frac{1}{(2\pi\hbar)^n}\int d\mu F(x)\,. \tag{35}$$

The basic property of the trace (specifying it up to multiplication on an arbitrary function of t) is vanishing on $*_t$-commutators, i.e.

$$\mathrm{Tr}_t([F,G]_t) = 0\,, \tag{36}$$

where $[F,G]_t \equiv F *_t G - G *_t F$ and at least one of the functions F and G vanishes on the infinity together with all its derivatives. Actually, the identity (36) follows from the stronger one: $\mathrm{Tr}_t(F *_t G) = \mathrm{Tr}_t(F\cdot G)$.

A *pure state* of a quantum mechanical system is described by a Wigner function $\rho(t,x)$ subject to the following conditions:

$$\rho *_t \rho = \rho\,, \qquad \mathrm{Tr}_t(\rho) = 1\,. \tag{37}$$

The quantum counterpart of the classical Liouville equation, governing the evalution of a quantum state, reads

$$i\hbar D_t\rho + [\rho, H]_t = 0\,, \tag{38}$$

where the extended time derivative D_t, defined by Eq. (24), can also be written as

$$D_tF \equiv \frac{\partial F}{\partial t} - \frac{1}{4i\hbar}\dot{\Omega}_{ij}(x^i *_t [x^j,\ F\ ]_t + [x^j,\ F\ ]_t *_t x^i)\,. \tag{39}$$

Note that Eq.(38) makes sense for an arbitrary (not necessary quadratic) Hamiltonians. For a constant Poisson bracket, (38) reproduces the usual von Neumann's equation for the symbol of statistical operator ρ. A simple direct calculation shows that the operator D_t differentiates the $*_t$-product:

$$D_t(F *_t G) = (D_tF) *_t G + F *_t (D_tG)\,. \tag{40}$$

As a consequence, every solution to Eq. (38), satisfying the idempotency condition (37) at some initial time moment, will satisfy this condition in all subsequent time moments. The specific choice of the integration measure in the definition of trace functional (35) provides the conservation of the normalization condition (37). Indeed, using the evolution equation for ρ and the property (36), one can find

$$\frac{d}{dt}\mathrm{Tr}_t(\rho(t)) = \left(\frac{d\ln\Delta}{dt} - \dot{\Omega}_{ij}\Pi^{ij}\right)\mathrm{Tr}_t(\rho(t)) = 0\,. \tag{41}$$

This property specifies the form of the trace functional up to an overall constant.

In the Schrödinger picture[3] the physical observables are considered to be chosen once and for all at some initial time moment, say $t = 0$. Only the quantum states evolve according to Eq. (38). The expectation value of an observable $F(x)$ at the time moment t relative to a state $\rho(t)$ is given by

$$\langle F\rangle^t_\rho = \mathrm{Tr}_t(F(x) *_t \rho(t,x))\,, \tag{42}$$

where the Wigner function ρ obeys Eqs.(37).

For quadratic Hamiltonians the equation (38) coincides with the modified Liouville equation (28). In that case, the evolution of an arbitrary quantum state ρ is governed by the linear first-order PDO

$$\frac{\partial\rho}{\partial t} - \frac{1}{2}x^i\dot{\Omega}_{ij}\{x^j,\rho\} + \{\rho, H\} = 0\,,$$

for which the initial classical equations (1) play the role of characteristics. So, we arrive at the following

Proposition 1. *Let $x^i(t) = \Gamma^i_j(t)x^j_0 + v^i(t)$ be the general solution to the classical equations of motion (1) with $x^i(t)|_{t=0} = x^i_0$, then the evolution of a quantum state $\rho(t,x)$ is given by the expression*

$$\rho(t,x) = \rho_0(\Lambda(t)[x - v(t)])\,,$$

where $\Lambda(t) = \Gamma^{-1}(t)$ and the initial state $\rho_0(x) = \rho(0,x)$ satisfies Eqs.(37) at $t = 0$.

[3]Of course, all the constructions can be straightforwardly reformulated in the Heisenberg picture.

Notice that the quantum evolution of linear systems is completely determined by the classical one and does not depend on any ambiguities concerning the choice of the quadratic action functional (2).

As we mentioned in Introduction the characteristic feature of dissipation is the presence of attractors, i.e. invariant subsets $A \in \mathbb{R}^{2n}$ to which all nearby trajectories converge. In other words, any classical state supported at a sufficiently small vicinity of A evolves to a state supported at A when $t \to \infty$. As the next proposition shows the similar phenomenon takes place at the quantum level as well (at least for linear systems with one-point attractor).

Proposition 2. *Given a system of differential equations $\dot{x} = A(t)x$ having the origin $x = 0$ as the global attractor, then each Wigner's function $\rho(t, x)$ defines a δ-shaped sequence*

$$\phi_t(x) = \frac{\Delta(t)}{(2\pi\hbar)^n}\rho(t, x)\,, \qquad \lim_{t\to\infty}\phi_t(x) = \delta(x)\,.$$

Proof. Since $x = 0$ is the global attractor $\lim_{t\to\infty} x(t) = 0$ for any solution $x(t)$. On the other hand, $x(t) = \Gamma(t)x_0$, and hence $\lim_{t\to\infty}\Gamma(t) = 0$. According to Proposition 1, for any compactly supported function $F(x) \in C_0^\infty(\mathbb{R}^{2n})$ we have

$$\int F(x)\phi_t(x)d^{2n}x = \frac{\Delta(t)}{(2\pi\hbar)^n}\int F(x)\rho_0(\Lambda(t)x)d^{2n}x = \frac{1}{(2\pi\hbar)^n}\int F(\Gamma(t)x)\rho_0(x)d^{2n}x\,. \tag{43}$$

Taking limit, we finally get

$$\langle F\rangle_\rho^\infty = \lim_{t\to\infty}\frac{1}{(2\pi\hbar)^n}\int F(\Gamma(t)x)\rho_0(x)d^{2n}x = F(0)\mathrm{Tr}_0(\rho_0) = F(0)\,.$$

Thus, $\phi_t(x)$ is a δ-shaped sequence.

4 Examples

By way of illustration let us consider the deformation quantization of two interesting dissipative systems: the damped linear oscillator and the radiating point charge in a homogeneous magnetic field. The corresponding classical dynamics has been discussed in Sec.2, including the time-dependent Poisson brackets. In both cases, dissipation has the form of a perturbation over a Hamiltonian system and we choose the basis quantum states as the eigen-states for the energy (and angular momentum) of the corresponding non-perturbed system. Using these states we then consider the evolution for the mean values of energy (and angular momentum) in the presence of dissipation.

4.1 Damped linear oscillator

Let $\rho(x, p)$ be the Wigner function describing a pure state of the harmonic oscillator ($\alpha = 0$) with a definite value of energy E. This amounts to saying that ρ solves the following eigen-value problem:

$$\begin{aligned} &H * \rho \;=\; \rho * H = E\rho\,,\\ &\rho * \rho = \rho\,, \qquad \bar{\rho} = \rho\,, \qquad \mathrm{Tr}(\rho) = 1\,, \end{aligned} \tag{44}$$

where

$$H = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) ,$$

and the $*$-product (33) is defined at $t = 0$ by the canonical Poisson brackets $\{p, x\} = 1$. Let us find ρ from (44) following [27]. The first (complex) equation in (44) is equivalent to the pair of the real ones

$$H\rho - \frac{\hbar^2}{4}\left(\omega^2\frac{\partial^2\rho}{\partial p^2} + \frac{\partial^2\rho}{\partial x^2}\right) = 2E\rho\,, \qquad \{H, \rho\} = 0\,. \tag{45}$$

The second equation implies that

$$\rho = \rho(H)\,. \tag{46}$$

Then the first equation yields

$$H\rho - \frac{\hbar^2}{4}(\rho'' H + \rho') = \frac{E}{\omega}\rho\,. \tag{47}$$

Introducing new variables

$$y = \frac{4}{\hbar\omega}H\,, \quad \rho = e^{-y/2} f\,,$$

one can bring Eq. (47) to the form

$$f'' y + (1 - y) f' + \left(\frac{E}{\hbar\omega} - \frac{1}{2}\right) f = 0\,. \tag{48}$$

This equation is satisfied by Laguerre's polynomials $L_n(y)$, provided

$$E = \hbar\omega\left(n + \frac{1}{2}\right)\,, \qquad n = 0, 1, 2, \ldots\,, \tag{49}$$

and these are known to exhaust all its solutions resulting in integrable Wigner's functions $\rho \in L^1(\mathbb{R}^2)$. Thus, any eigen-value (49) corresponds to the unique Wigner's function

$$\rho_n(H) = C_n \exp\left(-\frac{2H}{\hbar\omega^2}\right) L_n\left(\frac{4H}{\hbar\omega^2}\right) \tag{50}$$

The constant $C_n = (-1)^n 2/\omega$ is determined from the normalization condition (44). In Ref. [27], it was shown that the sequence $\{\rho_n\}$ defines a complete set of orthogonal projectors:

$$\rho_n * \rho_m = \delta_{mn}\rho_n\,, \qquad \delta(x - x')\delta(p - p') = \sum_{n=0}^{\infty} \rho_n(x, p)\rho_n(x', p')\,. \tag{51}$$

Now let us return to the damped linear oscillator with action (13). The time evolution of the mean energy H can easily be calculated using Proposition 1 and the formula (42). We have

$$\langle H\rangle^t_{\rho_n} = \frac{\Delta(t)}{(2\pi\hbar)^n}\int H(x)\rho_n(\Lambda(t)x)d^{2n}x = \frac{1}{(2\pi\hbar)^n}\int H(\Gamma(t)x)\rho_n(x)d^{2n}x\,, \tag{52}$$

where the initial state ρ_n is one of the states (50) and $\Gamma(t)$ solves Eq. (12) with $\Gamma(0) = 1$. Since

$$H(\Gamma(t)\xi) = e^{-\alpha t}H(\xi)\,, \qquad \xi = (x, p)^t\,, \tag{53}$$

we get

$$\langle H\rangle^t_{\rho_n} = \frac{e^{-\alpha t}}{2\pi\hbar}\int dpdx H * \rho_n = \frac{e^{-\alpha t}E_n}{2\pi\hbar}\int dpdx\rho_n = e^{-\alpha t}E_n\,. \tag{54}$$

As is seen the quantum evolution of the mean energy relative to the eigen-states ρ_n coincides exactly with the classical one: in both cases the energy decreases by exponential law.

4.2 Particle in homogeneous magnetic field

Consider the deformation quantization of the system (17). It is convenient to identify the complete set of observables with the energy H and the angular momentum L of the particle without friction ($A = 0$). Then the corresponding set of states with definite values of H and L is determined by the equations

$$\begin{aligned} H * \rho = \rho * H = E\rho\,, \qquad L * \rho = \rho * L = M\rho\,, \\ \rho * \rho = \rho\,, \qquad \bar{\rho} = \rho\,, \quad \mathrm{Tr}(\rho) = 1\,, \end{aligned} \tag{55}$$

where $E, M \in \mathbb{R}$ and

$$H = \frac{1}{2}\left(p - \frac{B}{2}y\right)^2 + \frac{1}{2}\left(q + \frac{B}{2}x\right)^2\,, \qquad L = py - qx\,. \tag{56}$$

The $*$-product is defined by the formula (33) at $t = 0$ w.r.t. the canonical Poisson brackets: $\{p, x\} = \{q, y\} = 1$ and the other brackets vanish.

In order to solve (55) we introduce the following linear canonical transformation $(p, x; q, y) \to (P, X; Q, Y)$:

$$\begin{aligned} P = p - \frac{B}{2}y\,, \qquad\qquad X = \frac{1}{B}\left(q + \frac{B}{2}x\right)\,, \\ Q = \frac{1}{B}\left(q - \frac{B}{2}x\right)\,, \quad Y = p + \frac{B}{2}y\,. \end{aligned} \tag{57}$$

together with the functions

$$H_1 = P^2 + B^2X^2\,, \qquad H_2 = Q^2 + B^2Y^2\,. \tag{58}$$

It easy to see that

$$H = H_1\,, \qquad L = B^{-1}(H_2 - H_1)\,. \tag{59}$$

Since H_1 and H_2 are nothing but the Hamiltonians of two independent harmonic oscillators, we have reduced the eigen-value problem (55) to the previous one (44). The Wigner functions solving Eqs.(55) are given by the (ordinary) products

$$\rho_{E,M} = \rho_n(H_1)\rho_l(H_2)\,, \tag{60}$$

where $\rho_m(H_{1,2})$ is defined by (50) with $\omega = B$, and the eigen-values E and M run trough the sets

$$E = \hbar B\left(n + \frac{1}{2}\right)\,, \qquad M = \hbar(l - n)\,, \qquad n, l = 0, 1, 2, ... \tag{61}$$

As is seen the eigen-values of the angular momentum L in a state with definite energy E are bounded from below by $-E/B + \hbar/2$.

Consider now the evolution of the mean values of H and L. Using the fundamental matrix of (19) and applying Proposition 1 we find

$$\begin{aligned} H(\xi, t) = H(\Gamma(t)\xi) = e^{2At}H(\xi)\,, \\ L(\xi, t) = L(\Gamma(t)\xi) = L(\xi) + \alpha(t)H(\xi) + \beta(t)K(\xi) + \gamma(t)N(\xi)\,, \end{aligned} \tag{62}$$

where $\xi = (x, p, y, q)^{\mathrm{t}}$ and

$$\alpha(t) = \frac{2A^2 e^{At}\cos(Bt) - A^2 + B^2 e^{2At}}{B(B^2 + A^2)}\,, \qquad K = PQ + XY\,,$$
$$\beta(t) = \frac{2A^2 e^{At}\cos(Bt) + 2Ae^{At}\sin(Bt)B - 2A^2}{B(B^2 + A^2)}\,, \qquad N = XQ - PY\,, \tag{63}$$
$$\gamma(t) = \frac{2A^2 e^{At}\cos(Bt)B - 2A^2 e^{At}\sin(Bt) - 2AB}{B(B^2 + A^2)}\,.$$

This gives immediately

$$\langle H\rangle^t_{E,M} = \frac{e^{2At}}{(2\pi\hbar)^2}\int d^4\xi H(\xi) * \rho_{E,M}(\xi) = \frac{e^{2At}E}{(2\pi\hbar)^2}\int d^4\xi \rho_{E,M}(\xi) = e^{2At}E\,. \tag{64}$$

The mean energy of the particle decreases by exponential low just as it behaves in the classical theory.

The invariance of the Wigner functions $\rho_{E,M}(\xi)$ under reversions in PX- and QY-planes suggests that $\langle K\rangle_{E,M} = \langle N\rangle_{E,M} = 0$, and hence

$$\langle L\rangle^t_{E,M} = \mathrm{Tr}_0(L(t) * \rho_{E,M}) = M - \alpha(t)E\,. \tag{65}$$

The same arguments show that

$$\langle\, x\rangle^t_{E,M} = \langle\, p\rangle^t_{E,M} = \langle\, y\rangle^t_{E,M} = \langle\, q\rangle^t_{E,M} = 0\,. \tag{66}$$

So, at each moment of time the measured values of coordinates and momenta relative to the state $\rho_{E,M}$ are equal to zero.

Taking successive limits $t \to \infty$ and $A \to 0$ in (65), we find that the limiting value of the angular momentum for a small friction $|A| \ll 1$ is given by $M - E/B$. Since $\beta(t) \to 0$, $\gamma(t) \to 0$ as $t \to \infty$, the same limiting value appears in the classical theory as well.

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